Dynamic programming using radial basis functions

Oliver Junge
Technische Universität München
oj@tum.de

joint work with *Alex Schreiber*

Abstract

We propose a discretization of the optimality principle in dynamic programming based on radial basis functions and Shepard's approximation method.

Consider a discrete time control system $x_{k+1} = f(x_k, u_k)$ with a continuous map $f : \Omega \times U \to \mathbb{R}^n$ on compact sets $\Omega \subset \mathbb{R}^n$, $U \subset \mathbb{R}^d$, $0 \in U$, as phase and control space, respectively. In addition, we are given a continuous *cost function* $c : \Omega \times U \to [0, \infty)$ and a compact *target set* $T \subset \Omega$ and we assume that $c(x, u)$ is bounded from below by a constant $\delta > 0$ for $x \notin T$ and all $u \in U$. Our goal is to design a *feedback law* $F : S \to U$, $S \subset \Omega$, $T \subset S$, that asymptotically stabilizes the system to the target set. To this end we employ the *optimality principle* [1],

$$V(x) = \inf_{u \in U}\{c(x, u) + V(f(x, u))\}, \quad x \in \Omega \backslash T, \tag{1}$$

where $V : \mathbb{R}^n \to [0, \infty]$ is the *(optimal) value function* which fulfills the boundary condition $V|_T = 0$ and, by definition, $V|_{\mathbb{R}^n \backslash \Omega} = \infty$.

Using the *Kružkov transform* [3] $V \mapsto v(\cdot) = \exp(-V(\cdot))$, (1) transforms to $v(x) = \sup_{u \in U}\left\{e^{-c(x,u)} v(f(x, u))\right\}, x \in \Omega \backslash T$, the boundary condition transforms to $v|_T = 1$ and we set $v|_{\mathbb{R}^n \backslash \Omega} = 0$. The right hand side of this fixed point equation yields the Bellman operator

$$\Gamma(v)(x) := \begin{cases} \sup_{u \in U}\left\{e^{-c(x,u)} \bar{v}(f(x, u))\right\} & x \in \Omega \backslash T, \\ 1 & x \in T, \\ 0 & x \in \mathbb{R}^n \backslash \Omega \end{cases}$$

on the Banach space $L^\infty = L^\infty(\mathbb{R}^n, \mathbb{R})$, where $\bar{v}(x) = v(x)$ for $x \in \mathbb{R}^n \backslash T$ and $\bar{v}(x) = 1$ for $x \in T$. By assumption on c, the Bellman operator $\Gamma : L^\infty \to L^\infty$ is a contraction and thus possesses a unique fixed point.

We are approximating v by an element of the *approximation space* $\mathcal{W} = \text{span}(w_1, \ldots, w_n)$, where

$$w_i(x) = \frac{\varphi_i(x)}{\sum_{j=1}^n \varphi_j(x)}, \quad i = 1, \ldots, n,$$

for $x \in \Omega$ and $w_i|_{\mathbb{R}^n \setminus \Omega} = 0$ and the $\varphi_i(x) = \varphi(\|x - x_i\|_2)$, $x \in \Omega$, are *radial basis functions* on the set $X = \{x_1, \ldots, x_n\} \subset \Omega$ of *nodes*, cf. [5, 2]. We assume the *shape function* φ to be nonnegative, so that the w_i are nonnegative, too. For some function $v : \mathbb{R}^n \to \mathbb{R}$, its *Shepard approximant* $Sv \in \mathcal{W}$ is given by, cf. [4], $Sv = \sum_{i=1}^n v(x_i) w_i$.

We now want to approximate the fixed point of Γ. To this end, we define the *Bellman-Shepard operator* to be $\tilde{\Gamma} := S \circ \Gamma : \mathcal{W} \to \mathcal{W}$. We show that the Shepard operator $S : (L^\infty, \|\cdot\|_\infty) \to (\mathcal{W}, \|\cdot\|_\infty)$ has norm 1 and that consequently the Bellman-Shepard operator $\tilde{\Gamma} : (\mathcal{W}, \|\cdot\|_\infty) \to (\mathcal{W}, \|\cdot\|_\infty)$ is a contraction and thus the iteration $v^{(k+1)} := S\left(\Gamma[v^{(k)}]\right)$ converges to the unique fixed point $\tilde{V} \in \mathcal{W}$ of $\tilde{\Gamma}$.

Numerical experiments illustrating the performance of the method will be given.

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