Necessary conditions in optimal control problems with integral equations of Volterra type

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Abstract

On a fixed time interval $[0,T]$ consider the control system:

$$x(t) = x(0) + \int_0^t f(t,s,x(s),u(s))ds,$$

where $x \in C^n[0,T]$, $u \in L^r_\infty[0,T]$. Among its solutions satisfying endpoint constraints $\eta(p) = 0$, $\varphi(p) \le 0$ with $p = (x(0), x(T))$, state and mixed constraints $\Phi(t, x(t)) \leqslant 0$, $F(t,x,u) \leqslant 0$, $G(t,x,u) = 0$, of dimensions $d(\eta)$, $d(\varphi)$, $d(\Phi)$, $d(F)$, $d(G)$, respectively, one should minimize an endpoint cost functional: $J = \varphi_0(p) \to \min$.

The data functions are assumed to be smooth. We assume that the endpoints of the reference state $\hat{x}(t)$ do not lie on the boundary of state constraints, i.e., $\Phi_k(0, \hat{x}(0)) < 0$, $\Phi_k(T, \hat{x}(T)) < 0$, for all $k = 1, \ldots, d(\Phi)$. Moreover, we assume that the mixed constraints are *regular*, i.e. at any point (t,x,u), the gradients with respect to u of all active inequality constraints and all equality constraints are positively–linearly independent.

Theorem 1. Let a process $(\hat{x}(t), \hat{u}(t))$ provide a weak minimum, i.e., a local minimum in the norm $||x||_C + ||u||_\infty$. Then $\exists\, (\alpha_0, \ldots, \alpha_{d(\varphi)}) \ge 0$, $(\beta_1, \ldots, \beta_{d(\eta)})$, measurable bounded functions $h_i(t) \ge 0$, $m_j(t)$, nondecreasing functions $\mu_k(t)$ continuous at 0 and T, and an $n-$vector function of bounded variation $\psi(t)$ also continuous at 0 and T, not all of them identically vanish, such that the following conditions hold:

endpoint complementary slackness: $\alpha_i\, \varphi_i(\hat{p}) = 0, \qquad i = 1, \ldots, d(\varphi),$
pointwise complementary slackness: $d\mu_k(t)\, \Phi_k(t, \hat{x}(t)) = 0$
and $h_i(t)\, F_i(t, \hat{x}(t), \hat{u}(t)) = 0 \qquad$ for all $k,\, i$;
transversality conditions $\psi(0) = l_{x(0)}\,, \quad \psi(T) = -l_{x(T)}\,,$ where
$l(x_0, x_T) = \left(\sum \alpha_i \varphi_i + \sum \beta_j \eta_j\right)(x_0, x_T)$ is the endpoint Lagrange function,
stationarity with respect to the control

$$\overline{H}_u(s, \hat{x}(s), \hat{u}(s)) = 0$$

and the adjoint (costate) equation

$$\dot{\psi}(s) = -\overline{H}_x(s, \hat{x}(s), \hat{u}(s)).$$

Here $f_x(t,s) = f_x(t, s, \hat{x}(s), \hat{u}(s)), \quad F_{iu}(s) = F_{iu}(s, \hat{x}(s), \hat{u}(s)),$ etc,
$\dot{\psi}$ denotes the generalized derivative, we use the convenient *modified Pontryagin function*

$$H(s, x, u) = \psi(s) f(s, s, x, u) + \int_s^T \psi(t)\, f_t(t, s, x, u)\, dt,$$

and the *extended modified Pontryagin function* $\overline{H}(s, x, u) =$

$$H(s, x, u) - \sum_i h_i(s)\, F_i(s, x, u) - \sum_j m_j(s)\, G_j(s, x, u) - \sum_k \dot{\mu}_k(s)\, \Phi(s, x).$$

To prove this, we consider our problem as a particular case of an abstract optimization problem in a Banach space with smooth equality constraints and possibly nonsmooth inequality constraints, we use the Lagrange multipliers rule for this problem (which can be obtained by the Dubovitskii–Milyutin approach [2]), and then apply the general optimality conditions to our specific problem. In this last step we use a theorem on the absence of singular components in the Lagrange multipliers at the mixed constraints [1, 2].

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References

[1] A.V. Dmitruk, *Maximum principle for a general optimal control problem with state and regular mixed constraints*, Computation Math. and Modeling, v. 4, no. 4 (1993), p. 364–377.

[2] A.A. Milyutin, A.V. Dmitruk, and N.P. Osmolovskii, *Maximum principle in optimal control*, Moscow State University, Faculty of Mechanics and Mathematics, Moscow, 2004 (in Russian), 168 p.