

# The heat equation associated to a time-optimal control problem linear in the control

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## Abstract

Let  $M$  be a two-dimensional smooth manifold and consider a pair of smooth vector fields  $X$  and  $Y$  on  $M$ . If the pair  $X, Y$  is Lie bracket generating, then the control system

$$(1) \quad \dot{q} = uX(q) + vY(q), \quad u^2 + v^2 \leq 1, \quad q \in M,$$

is completely controllable and the minimum-time function defines a continuous distance  $d$  on  $M$ . When  $X$  and  $Y$  are everywhere linear independent such distance is Riemannian and it corresponds to the metric for which  $(X, Y)$  is an orthonormal moving frame. The idea is to study the geometry obtained starting from a pair of vector fields which may become collinear. Under generic hypotheses, the set of points of  $M$  at which  $X$  and  $Y$  are parallel is a one-dimensional embedded submanifold of  $M$  (possibly disconnected). Metric structures that can be defined *locally* by a pair of vector fields  $(X, Y)$  through (1) are called almost-Riemannian structures.

In this talk we study the Laplace-Beltrami operator in almost-Riemannian structures. Under the assumption that the structure is 2-step Lie bracket generating, we prove that the Laplace-Beltrami operator is essentially self-adjoint and has discrete spectrum. As a consequence, a quantum particle cannot cross the singular set (i.e., the set where the vector fields become collinear) and the heat cannot flow through the singularity. This is an interesting phenomenon since when approaching the singular set all Riemannian quantities explode, but geodesics are still well defined and can cross the singular set without singularities.

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